ELSEVIER Journal of Geometry and Physics 30 (1999) 103–112

JOURNAL OF GEOMETRY AND PHYSICS

No-ghost theorem for a q-deformed open bosonic string

N. Mebarki [a,b,*], A. Boudine [a], H. Aissaoui [a], A. Benslama [a], A. Bouchareb [a], A. Maasmi [c,1]

[a] *Département de Physique Théorique, Constantine University, Constantine, Algeria*
[b] *Research Institute of Fundamental Physics, Kyoto University of Sakyo-ku-kyoto 606, Japan*
[c] *Département de Physique, Institut des Sciences exactes et de Technologie Centre Universitaire Larbi Tebessi, Tebessa, Algeria*

Received 30 March 1998

Abstract

We show that for the critical dimensions $D = 2 + 12(q+1)(0 \leq q \leq 1)$ the q-deformed Fock space is free of negative norm states (ghosts).

Subj. Class.: Strings
1991 MSC: 81T30
Keywords: Ghosts; Open bosonic strings

1. Introduction

There has been a lot of interest in the study of quantum groups [1–5] during the last decade. They appeared in the development of the quantum inverse method and the study of the Yang–Baxter equations [6,7]. Moreover, an interesting aspect of the quantum groups is related to the idea that symmetries under q-deformed groups or algebras (in the sense of the invariance under co-action) can be considered as the underlying principles for constructing sensitive theories.

The purpose of the this paper is to show that the q-deformed Fock space of an open bosonic string does not contain negative norm states for the space–time critical dimension $D = 2 + 12(q+1)$ $(0 < q \leq 1)$. In Section 2, we describe the formalism and in Section 3, we prove a No-ghost theorem and finally, in Section 4 we draw our conclusions.

* Corresponding author. Permanent address. Département de Physique Théorique, Constantine University, Constantine, Algeria
[1] On leave from Département de Physique Théorique, Constantine University.

PII: S0393-0440(98)00044-8

2. Formalism

The Nambu–Goto classical action of an open string is given by [8]

$$S = -\frac{1}{2\pi g a'} \int \mathrm{d}\tau\, \mathrm{d}\sigma [(\dot{x} - x')^2 - \dot{x}^2 x'^2]^{1/2}, \tag{1}$$

where τ, σ are dimensionless world-sheet parameters. Here α' is the string scale and $\dot{x}^\mu$ (resp. x'^μ) means $\partial x^\mu/\partial\tau$ (resp. $\partial x^\mu/\partial\sigma$). The general solution of equations of motion (in the orthonormal gauge) subject to the edge conditions

$$x'^\mu(\sigma = 0, \pi; \tau) = 0 \tag{2}$$

is

$$\ddot{x} - x'' = 0, \tag{3}$$

the general solution of this equation has an expression as follows:

$$x^\mu = q^\mu + 2\alpha' p^\mu + \mathrm{i}\sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [a_n^\mu \exp(-\mathrm{i}n\tau) + a_n^{\dagger\mu} \exp(\mathrm{i}n\tau)] \cos n\sigma,$$

where q^μ and p^μ are the string centre of mass coordinates and the momentum, respectively.

After q-deformation the physical states $\psi\rangle^q_{\text{phys.}}$ are subject to the Virasoro conditions

$$L_n^q |\psi^q_{\text{phys.}} = 0, \quad n \geq 1, \tag{4}$$

and

$$(L_0^q - \alpha_q(0))|\psi\rangle^q_{\text{phys.}} = 0, \tag{5}$$

where the q-deformed Virasoro generators L_n^q are given by

$$L_n^q = \frac{1}{4\alpha'} \sum_{-\infty}^{+\infty} : \alpha^\mu_{n-m} \alpha^\dagger_{n\mu} :_q , \tag{6}$$

where

$$\begin{aligned} \alpha_0^\mu &= 2\alpha' p^\mu, \\ \alpha_{-n}^\mu &= \sqrt{2\alpha' n} a_n^{\dagger\mu}, \quad n > 0, \\ \alpha_n^\mu &= \sqrt{2\alpha' n} a_n^\mu, \quad n > 0, \end{aligned} \tag{7}$$

and the dynamical variables [9–14] a_n^μ, $a_n^{\dagger\mu}$ satisfy the nonvanishing q-deformed commutation relations

$$[a_n^\mu, a_m^{\dagger\nu}]_q = \delta_{n,m} g^{\mu\nu} \tag{8}$$

and

$$[q^{\mu}, p^{\nu}]_q = \mathrm{i}g^{\mu\nu},$$

where

$$[a_n^{\dagger\mu}, a_m^{\nu}]_q = a_n^{\dagger\mu} a_m^{\nu} - [\delta^{\mu}_{\nu'}\delta^{\nu}_{\mu'} + (q-1)\delta_{n,m}\Lambda^{\mu\nu}_{\mu'\nu'}]a_m^{\nu'} a_n^{\dagger\mu} \tag{9}$$

and

$$[q^{\mu}, p^{\nu}]_q = q^{\mu}p^{\nu} - qp^{\nu}q^{\mu}. \tag{10}$$

Note that in Eq. (6), we have introduced a q-deformed normal ordering "$::_q$" which is defined as [10,11]

$$: a_n^{\mu} a_m^{\dagger\nu} :_q \equiv a_m^{\dagger\nu} a_n^{\mu} + (q-1)\delta_{n,m}\Lambda^{\mu\nu}_{\mu'\nu'} a_m^{\dagger\nu'} a_n^{\mu'}, \tag{11}$$
$$: a_m^{\dagger\nu} a_n^{\mu} :_q = a_m^{\dagger\nu} a_n^{\mu},$$

with

$$\Lambda^{\mu\nu}_{\mu'\nu'} \equiv \begin{cases} 1 & \text{if } \mu = \nu \text{ and } \mu' = \nu', \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

3. No-ghost theorem

An arbitrary state $|\psi\rangle^q_{\text{phys}}$ is called a physical state if it satisfies the constraints

$$L_n^q |\psi\rangle^q_{\text{phys}} = 0, \quad n > 0, \tag{13}$$

$$(L_0^q - \alpha_q(0))|\psi\rangle^q_{\text{phys}} = 0 \tag{14}$$

with

$$L_n^q = \frac{1}{4\alpha'} \sum_{-\infty}^{+\infty} : \alpha^{\mu}_{n-m} \alpha^{\dagger}_{n\mu} :_q . \tag{15}$$

It is worth to mention that $\alpha(0)$ is a c-number coming from the q-deformed normal ordering "$::_q$". The q-deformed Fock space is defined such that

$$a_n^{\mu}|0\rangle_q = 0, \quad n > 0, \qquad P^{\mu}|0\rangle_q = p^{\mu}|0\rangle_q \tag{16}$$

and

$${}_q\langle 0|0\rangle_q = 1. \tag{17}$$

A q-deformed state $|S\rangle_q$ which obeys

$$(L_0^q - \alpha_q(0))|S\rangle_q = 0 \tag{18}$$

is called spurious state if it is orthogonal to all physical states, i.e.

$$_q\langle S|\psi\rangle^q_{\text{phys}} = 0, \tag{19}$$

and it can always be written in the form

$$|S\rangle_q = \sum_{n>0}^{+\infty} L^q_{-n}|\Phi_n\rangle_q, \tag{20}$$

where $|\Phi_n\rangle_q$ is some q-deformed state that obeys

$$(L^q_0 - \alpha_q(0) + n)|\Phi_n\rangle_q = 0. \tag{21}$$

Actually, the infinite series in Eq. (20) can be truncated, since the L^q_n's for $n \geq 3$ can be represented as iterated commutators of L^q_{-1} and L^q_{-2}, e.g. $L^q_{-3}\alpha[L^q_{-1}, L^q_{-2}]_{q1,2}$ (see relations (32), (35) and (36)). So we can simply write a spurious state as

$$|S\rangle_q = L^q_{-1}|\Phi_1\rangle_q + L^q_{-2}|\Phi_2\rangle_q, \tag{22}$$

where $|\Phi_1\rangle_q$ and $|\Phi_2\rangle_q$ obey Eq. (21). Notice that the states of the form (22) are orthogonal to physical states $|\psi\rangle^q_{\text{phys}}$.

Now, if a q-deformed state $|\chi\rangle_q$ is both spurious and physical, then it has zero norm. Thus, these q-deformed states are orthonormal to all physical states, including themselves. We can construct states of this type by considering spurious state of the form

$$|\chi\rangle_q = (L^q_{-2} + \Omega_q(L^q_{-1})^2)|\Theta\rangle_q, \tag{23}$$

where Ω_q is a c-number (depending on q) and $|\Theta\rangle_q$ an arbitrary q-deformed state satisfying the constraints

$$L^q_n|\Theta\rangle_q = 0, \quad n > 0, \tag{24}$$

$$(L^q_0 - \alpha_q(0))(L^q_{-2} + \Omega_q(L^q_{-1})^2)|\Theta\rangle_q = 0, \tag{25}$$

and

$$(L_0^{-q-2}|\Theta\rangle_q = \Lambda_q|\Theta\rangle_q, \tag{26}$$

where Λ_q is a function to be determined and discussed later.

It is worth to mention that the constraint (25) yields to the following two relations:

$$L^q_0|\Theta\rangle_q = \frac{1}{q}[\alpha_q(0) - (1+q)]|\Theta\rangle_q \tag{27}$$

and

$$L^q_0|\Theta\rangle_q = \frac{1}{q^2}\left[\alpha_q(0) - \frac{(1+q)^2}{2}\right]|\Theta\rangle_q. \tag{28}$$

Consequently, we deduce that the q-deformed state $|\Theta\rangle_q$ satisfies

$$\left[L^q_0 + \frac{(1+q)}{2q}\right]|\Theta\rangle_q = 0 \tag{29}$$

and

$$\alpha_q(0) = \frac{(1+q)}{2}. \tag{30}$$

Now, in order that $|X\rangle_q$ have zero norm it must be physical, and in particular, it sould be annihilated by L_m^q for $m > 0$. Since it is trivially annihilated by L_m^q with $m > 3$, we need only consider to impose the conditions:

$$L_1^q|\Theta\rangle_q = 0, \qquad L_2^q|\Theta\rangle_q = 0. \tag{31}$$

After straightforward calculations and making use of the q-deformed Virasoro algebra [8–13]:

$$\begin{aligned}[L_n^q, L_m^q]_{q_{n,m}} &= L_m^{qn,m} L_m^{qn,m} - \Delta_{nm}^q L_m^{qn,m} L_n^{qn,m} \\ &= (n-m)L_{n+m}^q + C_{nm},\end{aligned} \tag{32}$$

where

$$\begin{aligned}C_{n,m} &= \delta_{0,m}\frac{(1-q)}{2} n L_n^q \\ &\quad + \delta_{n,-m}\left\{m\frac{(1-q)}{1+q}(3L_0^q - L_0^{-q-2}) + \frac{D-2}{12}m(m^2-1)\right\}\end{aligned} \tag{33}$$

$$\Delta_{nm}^q = q + \delta_{n,m}(1-q) \tag{34}$$

and

$$L_n^{qn,m} = -\frac{1}{4\alpha'}\sum_{l=-\infty}^{\infty} :\alpha_{n-l}^i \alpha_l^i :_{qn,m} \tag{35}$$

with

$$:\alpha_{n-l}^i \alpha_l^i :_{qn,m} = q^{-\delta_{n,m}\delta_{n-l,l}} :\alpha_{n-l}^i \alpha_l^i :_q \tag{36}$$

The q-deformed normal ordering "$::_q$" is defined as in Eqs. (11) and (12) together with Eqs. (9) and (10), and conditions (31) lead to

$$\begin{aligned}&\frac{q-5}{q} + \frac{D}{2} + \frac{2(1-q)}{1+q}\Lambda_q \\ &\quad + \frac{3}{2}\Omega_q\left[\frac{q-5}{q} + \frac{6(1-q)}{1+q}\Lambda_q\right] = 0, \\ &3 + \Omega_q\left\{-\frac{2}{q} + \frac{2(1-q)}{1+q}\Lambda_q - \frac{(1-q)^2}{1+q}\right\} = 0,\end{aligned} \tag{37}$$

which can be rewritten in the simplyfied form

$$x^2 + \sum_q x + Z_q = 0, \tag{38}$$

where

$$x = \frac{2(1-q)}{1+q}\Lambda_q,$$

$$\sum_q = \frac{q-7}{q} - \frac{27}{2} + \frac{D}{2} - \frac{(1-q)^2}{1+q}, \tag{39}$$

$$Z_q = \left(\frac{q-5}{q} + \frac{D}{2}\right)\left(\frac{2}{q} + \frac{(1-q)^2}{1+q}\right) + \frac{9}{2}\frac{(q-5}{q}.$$

Notice that in the ordinary case $q = 1$, $D = 26$ one has $x = 0$ and $Z_{q=1} = 0$ and therefore Eq. (38) is identically verified. For $q \neq 1$ and in order that Eq. (38) has a solution one has to have

$$\sum_q^2 -4Z_q \geq 0, \tag{40}$$

which implies that

$$\left[\frac{q-7}{q} - \frac{27}{2} + \frac{D}{2} - \frac{(1-q)^2}{1+q}\right]^2 \\ -4\left[\left(\frac{q-5}{q} + \frac{D}{2}\right)\left(\frac{2}{q} + \frac{(1-q)^2}{1+q}\right) + \frac{9}{2}\frac{(q-5)}{q}\right] \geq 0. \tag{41}$$

It is worth to mention that the critical dimension [10,11]

$$D_c = 2 + 12(q+1) \tag{42}$$

satisfied this inequality for $0 < q \leq 1$. In fact, as it is shown in Fig. 1, the function $f(q) = \sum_q^2 -4Z_q$ is positive for $q \in]0, 1]$.

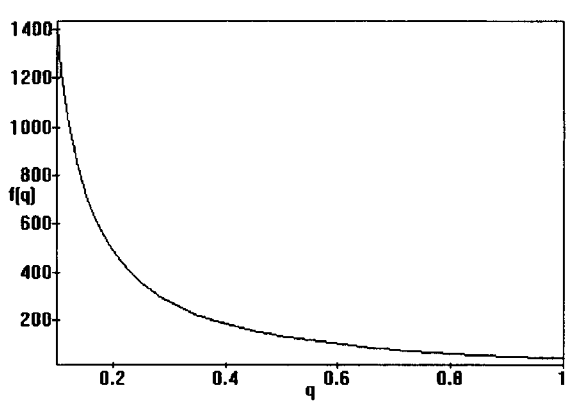

Fig. 1.

As an alternative way, to see explicitly that for the critical dimension (42), the q-deformed Fock space is free of negative norm states, let us consider the first three excited states of a q-deformed bosonic string:

$$|0\rangle_q, \qquad |\epsilon\rangle_q = \epsilon_\mu a_1^{\dagger\mu}|0\rangle_q, \qquad |\lambda,\theta\rangle_q = \frac{1}{\sqrt{2}}[\lambda_\mu a_1^{\dagger\mu} + \theta_{\mu\nu} a_1^{\dagger\mu} a_1^{\dagger}]|0\rangle_q. \tag{43}$$

By appplying the mass operator M^2:

$$M^2 = \frac{1}{\alpha'}[L_0^q - \alpha_q(0)] \tag{44}$$

on the vacuum state $|0\rangle_q$ and using relation (30) we obtain

$$M^2|0\rangle_q = -\frac{(1+q)}{\alpha'}|0\rangle_q. \tag{45}$$

Thus $|0\rangle_q$ is a tachyonic state for $q > -1$ and positive norm states for $q \leq -1$.

For the physical vectorial states $|\epsilon\rangle_q$, straightforward simplifications using relation (8) gives

$$_q\langle\epsilon|\epsilon\rangle_q = \frac{1}{q}\epsilon_\mu\epsilon^\mu. \tag{46}$$

Now, the q-deformed Virasoro condition

$$L_n^q|\epsilon\rangle_q = 0, \quad \text{for } n \geq 1, \tag{47}$$

is trivially verified for all values $n > 1$. However, for $n = 1$, one has to have

$$\epsilon_\mu P^\mu = 0. \tag{48}$$

Moreover, the mass operator M^2 applied on the physical state $|\epsilon\rangle_q$ implies that

$$-P^2 = \frac{1-q}{\alpha'} \tag{49}$$

with

$$P^2 = -P^{02} + P^{i2}, \quad i = \overline{1, D-1}. \tag{50}$$

Signature of the space–time is $(- + + + \cdots +)$.) Notice that in the rest frame, one can write

$$P^\mu = \left(\left\{\frac{1}{\alpha'}(1-q)\right\}^{1/2}, 0, 0\right), \tag{51}$$

assuming that $q \geq 1$, to avoid to have $|\epsilon\rangle_q$ as a tachyonic state. Combining Eqs. (48) and (51) one gets $\varepsilon^0 = 0$. Therefore, the norm (46) becomes

$$_q\langle\varepsilon|\varepsilon\rangle_q = \frac{\varepsilon^{i2}}{q}. \tag{52}$$

This clearly shows that ${}_q\langle\varepsilon|\varepsilon\rangle_q \geq 0$ providing that $q > 0$. Therefore, in order that $|\varepsilon\rangle_q$ will be a non-tachyonic state with a positive norm state one has to have:

$$0 < q \leq 1. \tag{53}$$

Now, regarding the state $|\lambda, \theta\rangle_q$ and with the use of q-deformed commutations relations one gets

$${}_q\langle\lambda, \theta|\lambda, \theta\rangle_q = \frac{1}{q}\left[\frac{\lambda^2}{2} + \theta_{\mu\nu}\theta^{\mu\nu}\right] \tag{54}$$

and

$$M^2|\lambda, \theta\rangle_q = \frac{3-q}{\alpha'}|\lambda, \theta\rangle_q = -p^2|\lambda, \theta\rangle_q. \tag{55}$$

For the Virasoro constraint:

$$L_n^q|\lambda, \theta\rangle_q = 0, \quad n > 0, \tag{56}$$

one can show that it is trivial except for $n = 1$ or 2. Straightforward calculations (for $n = 1$ and $n = 2$) lead to

$$\sqrt{2}(\lambda_\nu + \sqrt{4\alpha'}\theta_{\mu\nu}P^\mu)a_1^{+\nu}|0\rangle_q = 0 \tag{57}$$

and

$$-\frac{1}{\sqrt{2}}\left(\sqrt{4\alpha'}\lambda_\mu P^\mu + \frac{1}{q}\theta_{\mu\nu}\eta^{\mu\nu}\right)|0\rangle_q = 0. \tag{58}$$

As a result one has to have

$$\lambda_\nu + \sqrt{4\alpha'}\theta_{\mu\nu}P^\mu = 0 \tag{59}$$

and

$$\sqrt{4\alpha'}\lambda_\mu P^\mu + \frac{1}{q}\theta_{\mu\nu}\eta^{\mu\nu} = 0. \tag{60}$$

In the rest frame where p^μ is given by Eq. (51), relations (59) and (60) become

$$\begin{aligned} &\lambda_0 + \sqrt{4\alpha'}\theta_{00}P^0 = 0, \\ &\lambda_i + \sqrt{4\alpha'}\theta_{0i}P^0 = 0, \quad i = \overline{1, D-1}, \end{aligned} \tag{61}$$

and

$$\sqrt{4\alpha'}\lambda_0\left[\frac{3-q}{\alpha'}\right]^{1/2} - \frac{1}{q}\theta_{00} + \frac{1}{q}\sum_{i=1}^{D-1}\theta_{ii} = 0. \tag{62}$$

From Eqs. (54), (61) and (62) we deduce that

$$\begin{aligned} {}_q\langle\lambda, \theta|\lambda, \theta\rangle_q = \frac{1}{q}\Bigg\{ & 2\frac{(q-2)}{[2q(3-q)+1]^2}\left(\sum_{i=1}^{D-1}\theta_{ii}\right)^2 \\ & + (1-q)\sum_{i=1}^{D-1}\theta_{0i}^2 + \sum_{i=1}^{D-1}\theta_{ii}^2 + \sum_{i\neq j}^{D-1}\theta_{ij}^2\Bigg\}. \end{aligned} \tag{63}$$

Now, in order that the norm (63) is positive definite, independently of the parameters $\theta_{\mu\nu}$, one should have:

(a) $$(1-q) \geq 0, \quad \text{i.e.} \quad q \leq 1, \tag{64}$$

(b) $$\frac{2(q-2)}{[2q(3-q)+1]^2}\left(\sum_{i=1}^{D-1}\theta_{ii}\right)^2 + \sum_{i=1}^{D-1}\frac{\theta_{ii}}{q^2} \geq 0 \quad \forall\theta_{ii}. \tag{65}$$

Now, using the fact that $x^2 - x + 1 \geq 0\,\forall x$ (here in our case $x = \sum_{i=1}^{D-1}\theta_{ii}$), we obtain

$$\frac{-2(q-2)}{[2q(3-q)+1]^2}\left(1-\sum_{i=1}^{D-1}\theta_{ii}\right) + \sum_{i=1}^{D-1}\theta_{ii}^2 \geq 0, \tag{66}$$

using (67) direct simplifications lead to

$$D \leq 1 + 4(8q+1)^2 \tag{67}$$

with

$$0 < q \leq 1. \tag{68}$$

This means that

$$5 < D \leq 325. \tag{69}$$

Notice that the critical dimension $D_c = 2 + 12(q+1)$ where $0 < q \leq 1$ gives

$$14 < D_c \leq 26, \tag{70}$$

and consequently satisfies the double inequalities (70).

4. Conclusions

We conclude that the q-deformed open bosonic string critical dimension $D_c = 2 + 12(q+1)$ with $0 < q \leq 1$ guarantees that all the q-deformed states subject to the q-deformed Virasoro conditions (4) and (5) are physical, and consequently, the theory is free from negative norm states (ghosts).

Acknowledgements

We are grateful to Profs. Takesi Saito from Kyoto University and Yoshitaka Okumma from Chubu University for useful private communications. This work was supported by the ministry of education and research under contract N^{o}D2501/01/05/97.

References

[1] C.H. Oh, K. Singh, Generalized q-oscillators and their Hopf structure, J. Phys. A 27 (1994) 5907.

[2] N. Mebarki, S. Benrachi, H. Aissaoui, A. Boudine, The relationship between the paraquantization and deformation, Tr. J. Phys. 21 (1997) 235.
[3] S.V. Sabanov, Quantum and classical mechanics and q-deformed system, J. Phys. A, 26 (1993) 2583.
[4] M. Chaichain, A.P. Deemichev, q-Deformed path integral, Phys. Lett. B 320 (1994) 273.
[5] A.J. McFaclane, On q-analougues of the quantum harmonic oscillators and the quantum group $SU(2)_q$, J. Phys. A 22 (1989) 4581.
[6] M.A. Jimbo q-difference analougue of $U(g)$ and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985) 63.
[7] M.A. Jimbo q-analougue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter Equation 11 (1985) 247.
[8] J. Scherck, An introduction to the theory of dual models and strings, Rev. Mod. Phys. 47 (1975) 123.
[9] N. Mebarki, A. Boudine, H. Aissaoui, A. Benslama, A. Bouchareb, A. Maasmi, The q-deformed Wick theorem, Constantine University preprint, U.C. PUB 43, August 97.
[10] N. Mebarki, H Aissaoui, A. Boudine, A. Maasmi, The q-deformed Virasoro algebra, Czech. J. Phys. 47 (1997) 755.
[11] N. Mebarki, H. Aissaoui, A. Boudine, A. Maasmi, New critical dimensions for q-deformed bosonic string, Czech. J. Phys. 47 (1997) 761.
[12] N. Mebarki, A. Boudine, H. Aissaoui, A. Benslama, A. Bouchareb, The q-deformed Berry's phase for a generalized harmonic oscillator, Constantine University preprint, U.C. PUB 44, August 97.
[13] N. Mebarki, A. Boudine, H. Aissaoui, A. Benslama, A. Bouchareb, Path integral formalism of the q-deformed quantum mechanics, Constantine University preprint, U.C. PUB 45, August 97.
[14] N. Mebarki, A. Boudine, H. Aissaoui, A. Benslama, A. Bouchareb, in: Wigner matrix elements, 3j and 6j symbols and the casimir operator for $SU_q(2)$ guantum group, Constantine University preprint, U.C. PUB 46, August 97.